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On the Real *CF*-Method for Polynomial Approximation and Strong Unicity Constants

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We consider uniform polynomial approximation on $[-1, 1]$. For the class of functions which are analytic in an ellipse with foci ± 1 and sum of semiaxes greater than 8.1722..., we prove several asymptotic results on the best approximation. We describe the *CF*-approximation method and prove that, for our class of functions, the *CF*-approximation is "not far away" from the best one. With the help of this result we show a Kadec type result on the alternants and prove a conjecture of Poreda on the strong uniqueness constants. Also we prove a lemma on the distance between the best approximation and a "good" approximating polynomial. © 1988 Academic Press, Inc.

1. Introduction and Description of Results

We consider uniform approximation on the interval $[-1, 1]$ by polynomials, using the following notation:

$$C[-1, 1] = \{f : [-1, 1] \to \mathbb{R} : f \text{ continuous}\},$$

$$\|f\| = \|f\|_{[-1,1]} = \sup\{|f(x)| : x \in [-1, 1]\},$$

$$\Pi_n = \{p : p \text{ is a real polynomial of degree at most } n\},$$

$$e_n(f) = \inf\{\|f - p\| : p \in \Pi_n\}.$$

In the sequel we fix an $f \in C[-1, 1]$. The unique best approximation to f in Π_n will be denoted $p_n^*(f)$ or simply p_n^*, so that

$$\|f - p_n^*\| = e_n(f).$$

Freud [Fr] discovered that the operator $f \to p_n^*(f)$ is pointwise Lipschitz continuous. Later Cheney [Ch] gave an easy proof of that result based on the notion of strong unicity. He has shown that, for all $f, g \in C[-1, 1]$,

$$\frac{\|p_n^*(f) - p_n^*(g)\|}{\|f - g\|} \leqslant \frac{2}{\gamma_n(f)}, \tag{1.1}$$

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where $\gamma_n(f) > 0$ is the largest constant γ satisfying, for all $q \in \Pi_n$,

$$\|f - q\| \geqslant \|f - p_n^*(f)\| + \gamma \, \|q - p_n^*(f)\|. \tag{1.2}$$

Strong unicity has been discovered by Newman and Shapiro [N + Sh]. In a paper by Bartelt and McLaughlin [B + M] one can find the following characterization of the strong unicity constant $\gamma_n(f)$:

$$\gamma_n(f) = \min_{\substack{w \in \Pi_n \\ \|w\| = 1}} \max_{x \in E(r_n^*)} \operatorname{sgn}(r_n^*(x)) \cdot w(x), \tag{1.3}$$

where $r_n^* = f - p_n^*$ and, for $r \in C[-1, 1]$,

$$E(r) = \{x \in [-, 1] : |r(x)| = \|r\|\}.$$

The strong unicity constant (1.3) can also be defined on arbitrary Haar subspaces V of $C(X)$, X being a compactum. Then Π_n is replaced by V and the norm is to be taken on X.

In view of Eq. (1.1) it would be interesting to know if the sequence $(\gamma_n(f)^{-1})_{n \in \mathbb{N}}$ is bounded for any particular $f \in C[-1, 1]$. Poreda [Po] conjectured that for nonpolynomial f the sequence is unbounded or, equivalently,

$$\liminf_{n \to \infty} \gamma_n(f) = 0. \tag{1.4}$$

By the triangle inequality $\gamma_n(f) \leqslant 1$ for all $f \in C[-1, 1]$. For any polynomial $f \in \Pi_m$ one has $\gamma_n(f) = 1$ for $n \geqslant m$. Poreda's problem is still open, but we can solve it for a class of analytic functions. Let

$$\xi(\gamma) = \frac{2 - 3\gamma^2}{1 - 2\gamma^2} \cdot \gamma,$$

$$\lambda(\gamma) = \ln \gamma + \frac{8\xi(\gamma)}{1 - \xi(\gamma)^2},$$

and let γ^* be the smallest positive zero of λ. Then $\gamma^* \simeq 0.122366\ldots$ and, if we denote by E_r the closed ellipse with foci ± 1 and sum of semiaxes r, the following theorem will be proved.

1.1. THEOREM. *Let $f \in C[-1, 1]$ have an analytic continuation in E_r with $r > 1/\gamma^*$. Then (1.4) holds.*

The proof takes some time and will be given later. Of course the constant γ^* is of technical nature and most probably not optimal.

In the sequel we will make use of *CF*-approximations. This method has been described in detail by Hollenhorst [Ho], who refers to Darlington [Dar]. Further work on that topic has been done by Gutknecht and Trefethen [G + Tr]. The method can be described as follows.

Let f be analytic in an open set containing the closed ellipse $E_{1/\gamma}$. Then f is expandable in a series of Chebychev polynomials

$$f(z) = \sum_{n=0}^{\infty} \alpha_n T_n(z) \tag{1.5}$$

converging in that ellipse. One has

$$\limsup_{n \to \infty} |\alpha_n|^{1/n} < \gamma. \tag{1.6}$$

A theorem of Carathéodory and Fejér (see [Gol]) shows that there is a Blaschke product for all $m \in \mathbb{N}$, such that

$$\begin{aligned} q_{n,m}(z) &= \lambda_{n,m} \cdot z^{n+m} \cdot \prod_{\nu=1}^{m-1} \frac{1 - z_\nu z}{z - \overline{z_\nu}} \\ &= \sum_{\nu=-\infty}^{n} c_\nu z^\nu + \sum_{\nu=n+1}^{n+m} \alpha_\nu z^\nu \end{aligned} \tag{1.7}$$

with all its poles satisfying $|z_\nu| \leqslant 1$. The series converges outside the smallest circle with center 0 containing all the poles. Since the coefficients α_ν are real and the Blaschke product is uniquely determined, all the coefficients c_ν are real. We introduce now the linear isometric mapping $\hat{}$ from $C[-1, 1]$ to $C[0, \pi]$ defined by

$$\hat{f}(t) = f(\cos t). \tag{1.8}$$

It follows that

$$\hat{\Pi}_n = C_n = \left\{ q : q = \sum_{\nu=0}^{n} \beta_\nu \cos \nu t,\ \beta_\nu \in \mathbb{R} \right\} \tag{1.9}$$

and

$$\hat{f}(t) = \sum_{\nu=0}^{\infty} \alpha_\nu \cos \nu t = \operatorname{Re} \sum_{\nu=0}^{\infty} \alpha_\nu z^\nu \tag{1.10}$$

with $z = e^{it}$. Now one can define the CF-approximation to f by

$$\hat{p}_{n,m}(t) = \operatorname{Re}\left(\sum_{\nu=0}^{n} \alpha_\nu z^\nu - \sum_{\nu=-n}^{n} c_\nu z^\nu \right), \qquad z = e^{it}. \tag{1.11}$$

Then $p_{n,m}$ is in fact a real polynomial in Π_n. For the error function one has

$$\hat{r}_{n,m}(t) = \hat{f}(t) - \hat{p}_{n,m}(t) = \operatorname{Re}(q_{n,m}(z) - H_{n,m}(z) + R_{n+m}(z)), \quad (1.12)$$

where $z = e^{it}$,

$$H_{n,m}(z) = \sum_{\nu=-\infty}^{-n-1} c_\nu z^\nu, \quad (1.13)$$

$$R_{n+m}(z) = \sum_{\nu=n+m+1}^{\infty} \alpha_\nu z^\nu. \quad (1.14)$$

Under certain conditions $H_{n,m}$ and R_{n+m} are negligible and one has good information on the error function $r_{n,m}$. The following theorem due to Hollenhorst shows circumstances where an estimation of $H_{n,m}$ is possible.

1.2. THEOREM (Hollenhorst). *Suppose for* $\gamma < (\sqrt{13} - 1)/6$

$$|\alpha_{n+k}| \leqslant \gamma^{k-1} \cdot |\alpha_{n+1}|, \qquad \text{for} \quad k = 2, ..., m.$$

Then all the poles of $q_{n,m}$ *lie in*

$$B_{\xi(\gamma)} = \{z \in \mathbb{C} : |z| < \xi(\gamma)\}$$

and

$$|H_{n,m}(z)| \leqslant \frac{\gamma}{1 - 2\gamma^2} \cdot \frac{|\alpha_{n+1}|}{1 - \xi(\gamma)} \cdot \xi(\gamma)^{2n+1}.$$

Different proofs can be found in [Ho, Bl2, and Gr]. The function $\operatorname{Re} q_{n,m}(e^{it})$ has an alternating behaviour on $[0, \pi]$, with at least $n + 2$ alternation points. We can therefore use the following theorem to estimate the distance of the *CF*-approximation to the best one. The theorem is of interest on its own.

1.3. THEOREM. *Let*

$$-1 = x_0 < \cdots < x_{n+1} = 1$$

be points in $[-1, 1]$ *such that for* $q \in \Pi_n$ *and* $r = f - q$,

$$\operatorname{sgn} r(x_\nu) = (-1)^\nu \cdot \operatorname{sgn} r(x_0), \qquad \nu = 1, ..., n + 1.$$

Let

$$D = \max_{0 \leqslant \nu \leqslant n} (\arccos x_\nu - \arccos x_{\nu+1}),$$

$$d = \min_{0 \leqslant \nu \leqslant n} (\arccos x_\nu - \arccos x_{\nu+1}).$$

Then

$$\|p_n^* - q\| \leqslant (D/d)^{2n} \cdot (2n+1) \cdot (\|r\| - \min_{0 \leqslant \nu \leqslant n+1} |r(x_\nu)|).$$

The proof will be given in Section 2. With the help of this theorem we are able to prove the following result on *CF*-approximation.

1.4. THEOREM. *Suppose that* $\gamma < \gamma^*$ *and* $f \in C[-1, 1]$ *has an analytic continuation in the ellipse* $E_{1/\gamma}$. *Then we can choose a subsequence* $n_1 < n_2 < \cdots$ *such that for the coefficients in* (1.5),

$$|\alpha_{n_k+\mu}| \leqslant \gamma^{\mu-1} \cdot |\alpha_{n_k+1}|, \qquad \text{for all } \mu \in \mathbb{N}, k \in \mathbb{N}.$$

Taking $c_0 > c > 2$ *and*

$$m_k = \operatorname{int}\left(c_0 \cdot \frac{\ln n_k}{|\lambda(\gamma)|}\right),$$

where $\operatorname{int}(x)$ *denotes the greatest integer smaller than* x, *we have as* $k \to \infty$,

$$\|r_{n_k,m_k} - r_{n_k}^*\| = O(n_k^{-c+1} \cdot |\lambda_{n_k,m_k}|),$$

$$\|\hat{r}_{n_k}^*(t) - \operatorname{Re} q_{n_k,m_k}(e^{it})\|_{[0,\pi]} = O(n_k^{-c+1} \cdot |\lambda_{n_k,m_k}|),$$

$$\|p_{n_k,m_k} - p_{n_k}^*\| = \|r_{n_k,m_k} - r_{n_k}^*\| = O(n_k^{-c+1} \cdot \|r_{n_k}^*\|).$$

The proof will be given in Section 3. It should be noted that

$$|\lambda_{n,m}| = \max_{|z|=1} |q_{n,m}(z)| = \|q_{n,m}(e^{it})\|_{[0,\pi]}. \tag{1.15}$$

Compare also the results by Gutknecht and Trefethen [G + Tr]. With the help of these methods it is possible to derive results about the extremal set $E(r_n^*)$. For let us decompose $E(r_n^*)$ in nonempty sign components $E_{0,n}, ..., E_{j(n),n}$ such that

$$-1 \leqslant E_{0,n} < \cdots < E_{j(n),n} \leqslant 1,$$

$$E(r_n^*) = \bigcup_{\nu=0}^{j(n)} E_{\nu,n}, \tag{1.16}$$

$$\operatorname{sgn} r(E_{\nu,n}) = -\operatorname{sgn} r(E_{\nu-1,n}), \qquad \text{for} \quad \nu = 1, ..., j(n),$$

and choose any point

$$t_{\nu,n} \in E_{\nu,n}. \tag{1.17}$$

Then we can prove a result on $t_{\nu,n}$.

1.5. THEOREM. *Let the hypothesis of Theorem* 1.4 *be satisfied. With the notation of that theorem one has for* $j(n)$ *in* (1.16),

$$j(n_k) = n_k + 1, \qquad \text{for all} \quad k \geqslant K,$$

K big enough. For the points in (1.17) *one has as* $k \to \infty$,

$$\max_{0 \leqslant \nu \leqslant n_k + 1} \left| \arccos t_{\nu, n_k} - \frac{\nu\pi}{n_k + 1} \right| = O\left(\frac{\ln n_k}{n_k}\right),$$

$$\sup_{0 \leqslant \nu \leqslant n_k + 1} \; \sup_{s \in E_{\nu, n_k}} |\arccos t_{\nu, n_k} - \arccos s| = O(n_k^{-C}),$$

with $C = c/2 + 1/2$.

The constant C in this theorem can be made arbitrarily large. For our class of functions this theorem is an improvement over a theorem of Kadec [Kad], which holds for all $f \in C[-1, 1]$. The proof will be given in Section 4.

With the help of *CF*-approximation it is also possible to get results on the asymptotic behaviour of the strong unicity constant. We refer to [Gr]. (Compare also [H + Sw].)

2. INEQUALITIES ON STRONG UNCITY CONSTANTS

In order to prove Theorem 1.1 we need the following inequality for the strong unicity constant in (1.3).

2.1. LEMMA. *Suppose for the sign components in* (1.16) $j(n) = n + 1$ *and*

$$\sup_{0 \leqslant \nu \leqslant n+1} |\sup \arccos E_{\nu, n} - \inf \arccos E_{\nu, n}| \leqslant c.$$

Then

$$\gamma_n(f) \leqslant \frac{1}{n+1} + \frac{cn}{2}.$$

Proof. Let $\hat{\gamma}_n(\hat{f})$ denote the strong unicity constant to $\hat{f}$ approximated with respect to C_n. Then by Eq. (1.3) and since $\hat{}$ is isometric,

$$\gamma_n(f) = \hat{\gamma}_n(\hat{f}). \tag{2.1}$$

Let x_ν be the midpoint of the smallest interval containing $\arccos E_{\nu,n}$. By a theorem of Blatt [Bl1] there is a cosine polynomial $q \in C_n$ with $\|q\| = 1$ and

$$\max_{0 \leqslant \nu \leqslant n+1} (-1)^\nu q(x_\nu) \leqslant \frac{1}{n+1}. \tag{2.2}$$

(In fact Blatt shows that if the set $E(\hat{r}_n^*)$ consists of $n+2$ points then the strong unicity constant is $\leqslant 1/(n+1)$. This implies the existence of a normed q with (2.2).) By an inequality of Bernstein,

$$|q'(t)| \leqslant n, \qquad \text{for all} \quad t \in [0, \pi]. \tag{2.3}$$

Putting (2.2) and (2.3) together and using (1.3) yields our result. Q.E.D.

Our next aim is the proof of Theorem 1.3. For this we need a discrete version of the strong unicity constant. Let $n+2$ points,

$$-1 \leqslant x_0 < \cdots < x_{n+1} \leqslant 1, \tag{2.4}$$

in $[-1, 1]$ be given. Assume $E(r_n^*) = \{x_0, ..., x_{n+1}\}$. Then from (1.3) it is clear that $\gamma_n(f)$ does not depend on f. In fact

$$\gamma_n(x_0, ..., x_{n+1}) = \min_{\substack{w \in \Pi_n \\ \|w\| = 1}} \min_{0 \leqslant \nu \leqslant n+1} (-1)^\nu \cdot w(x_\nu). \tag{2.5}$$

This discrete strong unicity constant can be defined in the same way in C_n and also in the set of all trigonometric polynomials of degree $\leqslant n$, which we denote by $\mathbb{T}_n$. (In the latter case one has to take $2n+2 = \dim \mathbb{T}_n + 1$ points in $[0, 2\pi)$.) Its important role in our discussion stems from the following lemma.

2.2. Lemma. *Let $p \in \Pi_n$ be such that $r = f - p$ alternates in sign on a point set* (2.4). *Then*

$$\|p - p_n^*\| \leqslant \gamma_n(x_0, ..., x_{n+1})^{-1} \cdot (\|r\| - \min_{0 \leqslant \nu \leqslant n+1} |r(x_\nu)|).$$

Proof. For $\sigma = \pm 1$ we get

$$\begin{aligned} \|r\| \geqslant \|r_n^*\| &\geqslant \sigma \cdot (-1)^\nu \cdot r_n^*(x_\nu) \\ &= \sigma \cdot (-1)^\nu \cdot r(x_\nu) + \sigma \cdot (-1)^\nu \cdot (p - p_n^*)(x_\nu) \end{aligned}$$

for all $\nu = 0, ..., n+1$. Thus

$$\|r\| \geqslant \min_{0 \leqslant \nu \leqslant n+1} |r(x_\nu)| + \gamma_n(x_0, ..., x_{n+1}) \cdot \|p - p_n^*\|. \qquad \text{Q.E.D.}$$

To any point set (2.4) we associate another point set in $[0, 2\pi)$ by

$$t_{n+1-\nu} = \arccos x_\nu, \qquad \text{for} \quad \nu = 0, ..., n+1 \tag{2.6}$$

and

$$t_\nu = 2\pi - t_{2n+2-\nu}, \qquad \text{for} \quad \nu = n+2, ..., 2n+1. \tag{2.7}$$

Then

$$0 \leqslant t_0 < \cdots < t_{2n+1} < 2\pi. \tag{2.8}$$

If we denote by $\tilde{\gamma}_n(t_0, ..., t_{2n+1})$ the discrete strong unicity constant with respect to $\mathbb{T}_n$, defined as in (2.5), we have clearly

$$\gamma_n(x_0, ..., x_{n+1}) = \hat{\gamma}_n(t_0, ..., t_{n+1}) \geqslant \tilde{\gamma}(t_0, ..., t_{2n+1}). \tag{2.9}$$

Cline [C1] gave a characterisation for the strong unicity constant, which we write down for the trigonometric case (but analogous formulas hold for any Haar subspace).

$$\tilde{\gamma}_n(t_0, ..., t_{2n+1})^{-1} = \max_{0 \leqslant v \leqslant 2n+1} \|q_v\|_{[0, 2\pi]}, \tag{2.10}$$

where the t_v are as in (2.8), $q_v \in \mathbb{T}_n$, and

$$q_v(t_k) = (-1)^k, \qquad \text{for all} \quad k = 0, ..., 2n+1, k \neq v. \tag{2.11}$$

Define $L_{v,\mu} \in \mathbb{T}_n$ for $0 \leqslant v,\ \mu \leqslant 2n+1,\ v \neq \mu$, by

$$L_{v,\mu}(t_k) = \begin{cases} 1, & \text{for} \quad k = \mu \\ 0, & \text{for} \quad k \neq v, \mu. \end{cases} \tag{2.12}$$

Then

$$q_v = \sum_{\substack{\mu=0 \\ \mu \neq v}}^{2n+1} (-1)^\mu L_{v,\mu}. \tag{2.13}$$

We need a stronger version of (2.10).

2.3. Lemma. *Assume we have a point set as in* (2.8) *and define t_v for each $v \in \mathbb{Z}$ as a 2π-periodic continuation. Then with the trigonometric polynomials in* (2.11) *we have*

$$\tilde{\gamma}_n(t_0, ..., t_{2n+1})^{-1} = \max_{0 \leqslant v \leqslant 2n+1} \|q_v\|_{[t_{v-1}, t_{v+1}]} \tag{2.14}$$

and

$$|q_v(t)| = \sum_{\substack{\mu=0 \\ \mu \neq v}}^{2n+1} |L_{v,\mu}(t)|, \qquad \textit{for all} \quad t \in [t_{v-1}, t_{v+1}]. \tag{2.15}$$

Proof. Assume (2.14) is wrong. By (2.10) there must be a $t \in [t_\nu, t_{\nu+1}]$ such that for a $\mu \in \{0, ..., 2n+1\}$,

$$|q_\mu(t)| > \max\{|q_\nu(t)|, |q_{\nu+1}(t)|\}.$$

Of course $\mu \neq \nu$, $\mu \neq \nu+1$, and $t \in (t_\nu, t_{\nu+1})$.

First Case.

$$\operatorname{sgn} q_\mu(t) = (-1)^\nu = q_{\nu+1}(t_\nu) = q_\mu(t_\nu).$$

Since $q_{\nu+1}$ cannot have $2n+2$ zeroes in $[0, 2\pi)$ we have

$$\operatorname{sgn} q_{\nu+1}(t_{\nu+1}) = (-1)^\nu.$$

Thus

$$|q_\mu(t)| = (-1)^\nu q_\mu(t) > |q_{\nu+1}(t)| \geqslant (-1)^\nu q_{\nu+1}(t),$$

$$(-1)^\nu q_\mu(t_{\nu+1}) = -1 \leqslant |q_{\nu+1}(t_{\nu+1})| = (-1)^\nu q_{\nu+1}(t_{\nu+1}).$$

We get a zero of $q_\mu - q_{\nu+1}$ in $(t, t_{\nu+1}]$. But this function has $2n$ zeroes in t_k, $k \neq \mu$, and $k \neq \nu+1$. It has therefore $2n+1$ zeroes in $[t_\nu, t_\nu + 2\pi)$, which is a contradiction.

Second Case.

$$\operatorname{sgn} q_\mu(t) = (-1)^{\nu+1}.$$

In an analogous way one gets a zero in $[t_\nu, t)$ and a contradiction.

Because of (2.13) for a proof of (2.15) we need only show that $(-1)^\mu L_{\nu,\mu}$ has the same sign for all $\mu \neq \nu$ on $[t_{\nu-1}, t_{\nu+1}]$. But each $L_{\nu,\mu}$ has $2n$ sign changes at t_k, $k \neq \nu$, and $k \neq \mu$. Thus

$$\operatorname{sgn} L_{\nu,\mu}(t_\nu) = (-1)^{\mu-\nu+1} \operatorname{sgn} L_{\nu,\mu}(t_\mu) = (-1)^{\mu-\nu+1}.$$

Since $L_{\nu,\mu}$ has constant sign on $(t_{\nu-1}, t_{\nu+1})$ Eq. (2.15) follows. Q.E.D.

2.4. LEMMA. *Suppose we have a point set as in* (2.8) *and for* $0 < d \leqslant D$

$$d \leqslant t_{\nu+1} - t_\nu \leqslant D, \qquad \text{for} \quad \nu = 0, ..., 2n,$$

$$d \leqslant t_0 + 2\pi - t_{2n+1} \leqslant D.$$

Then

$$\tilde{\gamma}_n(t_0, ..., t_{2n+1}) \geqslant \left(\frac{d}{D}\right)^{2n} \cdot \frac{1}{2n+1}.$$

Equality holds for $d = D$.

Proof. The case $d = D = \pi/(n+1)$ may be found in [Cl]. (In fact it suffices to show $\|q_0\| = 2n+1$ for $t_\nu = \nu\pi/(n+1)$, $\nu = 0, ..., 2n+1$, because of (2.10) and the invariance of $\mathbb{T}_n$ under shifts. But

$$q_0(t) = \operatorname{Re} - (z^{-n} + \cdots + z^n)$$
$$= \operatorname{Re}\left(z^{n+1} - \frac{z^{2n+1} - 1}{z - 1} \cdot \frac{1}{z^n}\right)$$

with $z = e^{it}$.)

The rest of the lemma is proved by comparing with the equidistant case. It suffices to show

$$|q_0(0)| \leqslant (D/d)^{2n+1} \cdot (2n+1).$$

Using invariance properties of $\mathbb{T}_n$ one gets

$$\|q_\nu\|_{[t_{\nu-1}, t_{\nu+1}]} \leqslant (D/d)^{2n} \cdot (2n+1).$$

Then the result follows with Lemma 2.3.

We may assume now $t_0 > 0$. We construct equidistant points

$$0 < s_0 < \cdots < s_{2n+1} < 2\pi$$

with

$$\begin{aligned} t_\nu &\leqslant \frac{D}{\pi/(n+1)} \cdot s_\nu, && \text{for } \nu = 0, ..., n, \\ 2\pi - t_\nu &\leqslant \frac{D}{\pi/(n+1)} \cdot (2\pi - s_\nu), && \text{for } \nu = n+1, ..., 2n+1. \end{aligned} \tag{2.16}$$

This can be done by setting

$$s_0 = \frac{\pi/(n+1)}{t_0 + 2\pi - t_{2n+1}} \cdot t_0,$$
$$s_\nu = s_0 + \nu\pi/(n+1), \qquad \text{for } \nu = 1, ..., 2n+1.$$

To these equidistant points belong trigonometric poynomials $K_{\nu,\mu}$ defined in the same way (2.12) as $L_{\nu,\mu}$ with respect to the t_ν. We can use the following representation (similar for $K_{\nu,\mu}$):

$$L_{\nu,\mu}(t) = \prod_{r \neq \nu,\mu} \frac{\sin((t - t_r)/2)}{\sin((t_\mu - t_r)/2)}. \tag{2.17}$$

Due to (2.15) we need only show $|L_{0,\mu}(0)| \leqslant (D/d)^{2n} \cdot |K_{0,\mu}|$ for all $\mu \neq 0$. By (2.17) we can reduce this to

$$\frac{|\sin(t_r/2)|}{|\sin((t_r - t_\mu)/2)|} \leqslant \frac{D}{d} \cdot \frac{|\sin(s_r/2)|}{|\sin((s_r - s_\mu)/2)|}$$

for $r = 1, ..., 2n+1$, $r \neq \mu$.

One can show

$$|\sin y| \leqslant \alpha \cdot |\sin x|, \qquad \text{for all} \quad x \in [0, \pi/2), \alpha \geqslant 1, y \in [0, \alpha x]. \tag{2.18}$$

From (2.16) and (2.18) one gets for $r = 1, ..., n$,

$$|\sin(t_r/2)| \leqslant \frac{D}{\pi/(n+1)} \cdot |\sin(s_r/2)| \tag{2.19}$$

since $s_r \in [0, \pi]$. Since $2\pi - s_r \in [0, \pi]$ for $r = n+1, ..., 2n+1$, one gets (2.19) for those r, again using (2.18).

Now take $r \neq \mu$.

First Case. $|t_\mu - t_r| \leqslant |s_\mu - s_r|$. Then

$$|t_\mu - t_r| \geqslant |\mu - r| \cdot d,$$
$$|s_\mu - s_r| = |\mu - r| \cdot \pi/(n+1).$$

Since sin is a concave function on $[0, \pi]$ we get

$$\left| \sin \frac{t_\mu - t_r}{2} \right| \geqslant \frac{|t_\mu - t_r|}{|s_\mu - s_r|} \cdot \left| \sin \frac{s_\mu - s_r}{2} \right|$$
$$\geqslant \frac{d}{\pi/(n+1)} \cdot \left| \sin \frac{s_\mu - s_r}{2} \right|. \tag{2.20}$$

Second Case. $|t_\mu - t_r| \geqslant |s_\mu - s_r|$. Now

$$|(2\pi + t_r) - t_\mu| \leqslant |(2\pi + s_r) - s_\mu|$$

and

$$|(2\pi + t_r) - t_\mu| \geqslant (2n + 2 + r - \mu) \cdot d,$$
$$|(2\pi + s_r) - s_\mu| = (2n + 2 + r - \mu) \cdot \pi/(n+1).$$

In the same way as in the first case one gets (2.20). By (2.19) and (2.20) the proof is complete. Q.E.D.

Proof of Theorem 1.3. The proof of this theorem is now a consequence of Lemmas 2.2, 2.4, and inequality (2.9).

3. Results on the *CF*-Method

In this section we aim to prove Theorem 1.4. At first we need a trivial result.

3.1. Lemma. *Let* $(\alpha_n)_{n\in\mathbb{N}}$ *be a sequence in* $\mathbb{C}$ *with*

$$\limsup_{n\to\infty} |\alpha_n|^{1/n} < \gamma < 1.$$

Then there is a subsequence $n_1 < n_2 < \cdots$ *such that*

$$|\alpha_{n_k+\mu}| \leqslant \gamma^{\mu-1} \cdot |\alpha_{n_k+1}|, \qquad \textit{for all} \quad k\in\mathbb{N}, \mu\in\mathbb{N}.$$

Proof. Take $\limsup |\alpha_n|^{1/n} < c < \gamma$. Then $|\alpha_n| < c^n < \gamma^n$ for $n \geqslant N$, N big enough. Thus $\lim |\alpha_n|/\gamma^n = 0$. Therefore we can choose a subsequence (n_k) with

$$\frac{|\alpha_{n_k+\mu}|}{\gamma^{n_k+\mu}} < \frac{|\alpha_{n_k+1}|}{\gamma^{n_k+1}}$$

for all $k\in\mathbb{N}$, $\mu\in\mathbb{N}$. Q.E.D.

Thus the first result of Theorem 1.4 follows from observation (1.6). We fix now any such subsequence, $c > 2$ and m_k due to Theorem 1.4. Since $\gamma < \gamma^* < (\sqrt{13}-1)/6$ we have $\gamma < \xi(\gamma) < 1$. By Theorem 1.2 we get

$$\|H_{n_k,m_k}\|_S = O(\xi(\gamma)^{2n_k+1} \cdot |\alpha_{n_k+1}|), \tag{3.1}$$

where we denote by S the set $\{z\in\mathbb{C} : |z| = 1\}$ and take the norm on this set. Also we have

$$\|R_{n_k+m_k}\|_S = O(\gamma^{m_k} \cdot |\alpha_{n_k+1}|). \tag{3.2}$$

Thus by Eq. (1.12),

$$\|\hat{r}_{n_k,m_k}(t) - \operatorname{Re} q_{n_k,m_k}(e^{it})\|_{[0,2\pi)} = O(\gamma^{m_k} \cdot |\alpha_{n_k+1}|). \tag{3.3}$$

Since the Blaschke product (1.7) has constant modulus $|\lambda_{n,m}|$ on S we get by the Cauchy integral formula

$$|\alpha_{n_k+1}| \leqslant |\lambda_{n_k,m_k}| = \|q_{n_k,m_k}\|_S = \|\operatorname{Re} q_{n_k,m_k}(e^{it})\|_{[0,2\pi)}. \tag{3.4}$$

Thus

$$\|\hat{r}_{n_k,m_k}(t) - \operatorname{Re} q_{n_k,m_k}(e^{it})\|_{[0,\pi]} = O(\gamma^{m_k} \cdot \|\operatorname{Re} q_{n_k,m_k}(e^{it})\|_{[0,\pi]}). \tag{3.5}$$

(The functions here are symmetric to π. Thus the norms on $[0, \pi]$ and on $[0, 2\pi)$ are equal.) This motivates the following lemma.

3.2. LEMMA. *Let $q = q_{n,m}$ be as in Eq. (1.7) and let all coefficients be real. We assume for the poles of q,*

$$|z_\nu| < \xi < 1, \qquad \text{for} \quad \nu = 1, ..., m-1, \tag{3.6}$$

and

$$(m-1) \cdot \frac{2\xi}{\xi - 1} < (n+1). \tag{3.7}$$

Let $r: S \to \mathbb{C}$ be a continuous function with

$$\|\operatorname{Re}(r-q)\|_S \leqslant \|q\|_S \cdot \varepsilon \tag{3.8}$$

for some $0 \leqslant \varepsilon \leqslant 1/2$. Then there exist compact sets $K_0, ..., K_{n+1}$ with

$$0 \leqslant K_0 < \cdots < K_{n+1} \leqslant \pi, \tag{3.9}$$

$$\bigcup_{\nu=0}^{n+1} K_\nu = \{t \in [0, \pi] : |\operatorname{Re} r(e^{it})| \geqslant \|q\|_S \cdot (1-\varepsilon)\}, \tag{3.10}$$

$$\operatorname{sgn} r(K_\nu) = (-1)^\nu \cdot \operatorname{sgn} r(K_0), \qquad \text{for} \quad \nu = 1, ..., n+1. \tag{3.11}$$

These sets are uniquely determined. If $\varepsilon = 0$ then $K_\nu = \{t_\nu\}$ for $\nu = 0, ..., n+1$. We have $0 \in K_0$, $\pi \in K_{n+1}$, and

$$\sup K_\nu - \inf K_\nu \leqslant 2 \cdot \sqrt{6\varepsilon}/(n+1-(m-1) \cdot 2\xi/(1-\xi)), \tag{3.12}$$

$$\sup K_\nu - \inf K_{\nu-1} \leqslant (\pi + 2 \cdot \sqrt{6\varepsilon})/(n+1-(m-1) \cdot 2\xi/(1-\xi)), \tag{3.13}$$

$$\inf K_\nu - \sup K_{\nu-1} \geqslant (\pi - 2 \cdot \sqrt{6\varepsilon})/(n+1+(m-1) \cdot 2\xi/(1+\xi)), \tag{3.14}$$

for $\nu = 1, ..., n+1$ ((3.12) also for $\nu = 0$).

Proof. At first we note that there is a continuously differentiable argument function ψ_ν on $[0, 2\pi]$ with

$$e^{i \cdot \psi_\nu(t)} = \frac{1 - z_\nu z}{z - \overline{z_\nu}}, \qquad \text{for} \quad t \in [0, 2\pi], z = e^{it} \tag{3.15}$$

$(\nu = 1, ..., m-1)$. We define then an argument function for q by

$$\arg q(e^{it}) = \arg \lambda_{n,m} + (n+m) \cdot t + \sum_{\nu=1}^{m-1} \psi_\nu(t), \qquad t \in [0, 2\pi]. \tag{3.16}$$

One has

$$|\psi_\nu'(t)| = \left|\frac{d}{dt} e^{i \cdot \psi_\nu(t)}\right|$$
$$= \left|\frac{d}{dz} \frac{1 - z_\nu z}{z - \overline{z_\nu}}\right|$$
$$= \frac{1 - |z_\nu|^2}{|z - \overline{z_\nu}|^2}.$$

Thus ψ_ν' has the same sign for all $\nu \in [0, 2\pi]$ and clearly this sign is negative. Thus by (3.6),

$$-\frac{1-\xi}{1+\xi} \geqslant \psi_\nu'(t) \geqslant -\frac{1+\xi}{1-\xi} \tag{3.17}$$

for $t \in [0, 2\pi]$. By (3.16) we get

$$\frac{d}{dt} \arg q(e^{it}) \leqslant n + m - (m-1) \cdot (1-\xi)/(1+\xi)$$
$$= n + 1 + (m-1) \cdot 2\xi/(1+\xi), \tag{3.18}$$
$$\frac{d}{dt} \arg q(e^{it}) \geqslant n + m - (m-1) \cdot (1+\xi)/(1-\xi)$$
$$= n + 1 - (m-1) \cdot 2\xi/(1-\xi). \tag{3.19}$$

With our assumption (3.7) we see that $\arg q(e^{it})$ is increasing on $[0, 2\pi]$. Since

$$\gamma(t) = q(e^{it}), \qquad t \in [0, 2\pi],$$

has winding number $n+1$ (count the number of zeroes and poles of q in the unit circle) and

$$\overline{q(\bar{z})} = q(z), \qquad \text{for} \quad z \in S,$$

we get intervals $I_0, ..., I_{n+1}$ with $0 \in I_0$, $\pi \in I_{n+1}$, and

$$0 \leqslant I_0 < \cdots < I_{n+1} \leqslant \pi, \tag{3.20}$$
$$\bigcup_{\nu=0}^{n+1} I_\nu = \{t\colon |\text{Re}\, q(e^{it})| \geqslant \lambda \cdot (1 - 2\varepsilon),\ t \in [0, \pi]\}, \tag{3.21}$$
$$\text{sgn Re}\, q(I_\nu) = (-1)^\nu \cdot \text{sgn Re}\, q(I_0), \qquad \nu = 1, ..., n+1. \tag{3.22}$$

K_ν is then uniquely determined by

$$K_\nu = \{t \in I_\nu : |\mathrm{Re}\, r(e^{it})| \geqslant \|q\|_S \cdot (1-\varepsilon)\},$$

$\nu = 0, ..., n+1$. Since $|q(1)| = |q(-1)| = |q(-1)| = \|q\|_S$ we have $0 \in K_0$, $\pi \in K_{n+1}$, and clearly K_ν satisfy (3.9)–(3.11).

(3.12)–(3.14) are now consequences of the mean value formula. We work it out for (3.12).

$$\begin{aligned}\sup K_\nu - \inf K_\nu &\leqslant \sup I_\nu - \inf I_\nu \\ &\leqslant (\arg q(e^{i\cdot \sup I_\nu}) - \arg q(e^{i\cdot \inf I_\nu}))/(n+1-(m-1)\cdot 2\xi/(1-\xi)) \\ &\leqslant 2 \arccos(1-2\varepsilon)/(n+1-(m-1)\cdot 2\xi/(1-\xi)) \\ &\leqslant 2\cdot \sqrt{6\varepsilon}/(n+1-(m-1)\cdot 2\xi/(1-\xi)).\end{aligned}$$

The statement on $\varepsilon = 0$ is clear by the monotonicity of $\arg q(e^{it})$. Q.E.D.

Proof of Theorem 1.4. We have already chosen the subsequence $n_1 < n_2 < \cdots$ and the m_k. For

$$\varepsilon = 0, \qquad \xi = \xi(\gamma), \qquad q = q_{n_k, m_k}, \qquad r(e^{it}) = \hat{r}_{n_k, m_k}(t),$$

we apply Lemma 3.2. Since $m_k = o(n_k)$ as $k \to \infty$, by (3.5) and Theorem 1.2 the hypothesis of Lemma 3.2 is satisfied for $k \geqslant N$, N big enough. We get for the points $t_\nu = t_{\nu,k}$ $(\nu = 0, ..., n_k + 1)$,

$$\begin{aligned}\pi/(n_k + 1 + (m_k - 1)\cdot 2\xi/(1+\xi)) &\leqslant |t_{\nu+1,k} - t_{\nu,k}| \\ &\leqslant \pi/(n_k + 1 + (m_k-1)\cdot 2\xi/(1-\xi)).\end{aligned}$$

Also by (3.5)

$$\begin{aligned}&\|\hat{r}_{n_k,m_k}\|_{[0,\pi]} - \min_{0 \leqslant \nu \leqslant n_k+1} |\hat{r}_{n_k,m_k}(t_{\nu,k})| \\ &\quad\leqslant 2\cdot \|\mathrm{Re}(r-q)\|_S \\ &\quad= O(\gamma^{m_k}\cdot \|q_{n_k,m_k}\|_S)\end{aligned}$$

and $\hat{r}_{n_k,m_k}$ alternates in sign in the points $t_{\nu,k}$. Applying Theorem 1.3 we get

$$\begin{aligned}&\|r_{n_k,m_k} - r^*_{n_k}\|_{[-1,1]} \\ &\quad\leqslant \left(\frac{n_k + 1 + (m_k-1)\cdot 2\xi/(1+\xi)}{n_k+1-(m_k-1)\cdot 2\xi/(1-\xi)}\right)^{2n_k} \cdot (2n_k+1)\cdot O(\gamma^{m_k}\cdot |\lambda_{n_k,m_k}|).\end{aligned} \tag{3.23}$$

(By (1.15) $\|q_{n_k,m_k}\|_S = |\lambda_{n_k,m_k}|$.) One has

$$\left(1+\frac{m_k-1}{n_k+1}\cdot\frac{2\xi}{1+\xi}\right)^{2n_k} \leqslant e^{m_k\cdot 4\xi/(1+\xi)},$$

$$\left(1-\frac{m_k-1}{n_k+1}\cdot\frac{2\xi}{1-\xi}\right)^{-2n_k} = O(e^{m_k\cdot(4+\varepsilon)\xi/(1-\xi)})$$

for any $\varepsilon>0$. Thus as $k\to\infty$

$$\begin{aligned}\|r_{n_k,m_k}-r^*_{n_k}\|_{[-1,1]} &= O(e^{m_k\cdot(8+\varepsilon)\xi/(1-\xi^2)}\cdot n_k\cdot\gamma^{m_k}\cdot|\lambda_{n_k,m_k}|)\\ &= O(e^{c_0/|\lambda(\gamma)|\cdot(8+\varepsilon)\xi/(1-\xi^2)\cdot\ln(n_k)}\\ &\quad\cdot n_k\cdot\gamma^{c_0/|\lambda(\gamma)|\cdot\ln(\gamma)}\cdot|\lambda_{n_k,m_k}|)\\ &= O(n_k^{c_0/|\lambda(\gamma)|\cdot(\ln(\gamma)+(8+\varepsilon)\xi/(1-\xi^2))+1}\cdot|\lambda_{n_k,m_k}|)\\ &= O(n_k^{-c+1}\cdot|\lambda_{n_k,m_k}|).\end{aligned}$$

This is the first inequality in Theorem 1.4. The second one follows with (3.5). The third one is a consequence of the first two. Q.E.D.

4. Proof of Theorem 1.1 and Theorem 1.5

We make use of Lemma 3.2 with

$$r(e^{it}) = \hat{r}^*_{n_k}(t), \qquad q = q_{n_k,m_k}.$$

Set

$$\varepsilon = \varepsilon_k = \|\hat{r}^*_{n_k}(t) - \operatorname{Re} q_{n_k,m_k}(e^{it})\|_{[0,\pi]}/|\lambda_{n_k,m_k}|. \tag{4.1}$$

Then by Theorem 1.4 we have

$$\varepsilon_k = O(n_k^{-c+1}). \tag{4.2}$$

The hypothesis of Lemma 3.2 is then fulfilled for $k\geqslant N$, N big enough (applying Theorem 1.2 again). If we denote the sets of Lemma 3.2 with $K_{v,k}$, we have for $k\geqslant N$,

$$\max_{0\leqslant v\leqslant n_k+1}(\sup K_{v,k} - \inf K_{v,k}) = o(n_k^{(-c+1)/2-1}). \tag{4.3}$$

For the extremal sign components in (1.16) we have

$$\arccos E_{v,k} \subseteq K_{v,k}, \qquad \text{for} \quad v=0,\ldots,j(n_k)=n_k+1 \text{ and } k\geqslant N. \tag{4.4}$$

Theorem 1.1 follows now from (4.2) and (4.4) if we apply Lemma 2.1. The second equality in Theorem 1.5 is equivalent to (4.2). To prove the first equality set

$$s_{\nu,n} = \nu\pi/(n+1), \tag{4.5}$$

$$h_k = \max_{1 \leqslant \nu \leqslant n_k+1} |\arccos t_{\nu+1,k} - \arccos t_{\nu,k}|, \tag{4.6}$$

$$H_k = \max_{0 \leqslant \nu \leqslant n_k+1} |\arccos t_{\nu+1,k} - s_{\nu,n_k}|. \tag{4.7}$$

Then we proceed as above and get for $\nu = 0, ..., n+1$,

$$\arccos t_{\nu,k} \in K_{\nu,k}. \tag{4.8}$$

Using Lemma 3.2 we get

$$(\pi - 2\cdot\sqrt{6\varepsilon_k})/(n_k + 1 + (m_k - 1)\cdot 2\xi/(1+\xi)) \leqslant h_k \leqslant (\pi + 2\cdot\sqrt{6\varepsilon_k})/(n_k + 1 - (m_k - 1)\cdot 2\xi/(1-\xi)). \tag{4.9}$$

Thus

$$|h_k - \pi/(n_k+1)| = O(\ln n_k/n_k^2). \tag{4.10}$$

Then clearly for $n+1 \geqslant \nu > \mu \geqslant 0$,

$$|\arccos t_{\nu,k} - \arccos t_{\mu,k} - (\nu - \mu)\cdot \pi/(n_k+1)| = O(\ln n_k/n_k). \tag{4.11}$$

Due to (4.3) and (4.8)

$$|\arccos t_{\nu,0}| = O(\ln n_k/n_k). \tag{4.12}$$

From (4.11) and (4.12) we get as $k \to \infty$,

$$H_k = O(\ln n_k/n_k). \qquad \text{Q.E.D.}$$

Note

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